On the Notation of Field Equations of Electrodynamics

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Maxwell's equations are the cornerstone in electrodynamics. Despite the fact that these equations are more than hundred years old, they still are subject to changes in recent publications. To get an impression over the historical development of Maxwell's equations, the equation systems in different notations are summarized.

Introduction

The complete set of the equations of James Clerk MAXWELL[15] are known in electrodynamics since 1865. These have been defined for 20 field variables. Later Oliver HEAVISIDE[11] and William GIBBS[23] have transformed this equations into the today's most used notation with vectors. This has not been happened without ‚background noise'[3], then at that time many scientists – one of them has been MAXWELL himself – was convinced, that the correct notation for electrodynamics must be possible with quaternions[5] and not with vectors. A century later EINSTEIN introduced Special Relativity and since then it was common to summarize MAXWELL's equations with four-vectors.

The search at magnetic monopoles has not been coming to an end, since DIRAC[4] introduced a symmetric formulation of MAXWELL's equations without using imaginary fields. But in this case the conclusion from Special Theory of Relativity, that the magnetic field originates from relative motion only, can not be hold anymore.

The non-symmetry in MAXWELL's equations of the today's vector notation may have dissatisfied many scientists intuitively. That could be the reason, why they published an extended set of equations. Sometime these extensions have been introduced for specific applications only. This essay summarizes the main different notation forms of MAXWELL's equations.

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Maxwell's Equations

The Original Equations

With the knowledge of fluid mechanics MAXWELL[15] has introduced the following 20 equations for the electromagnetic fields. The equations on the left correspond with the original text, the equations on the right correspond with today's well known vector notation:

$$\begin{aligned} p' &= p + \frac{d f}{dt} \\ q' &= q + \frac{d g}{dt} \\ r' &= r + \frac{d h}{dt} \end{aligned} \quad \rightarrow \quad \begin{aligned} J_1 &= j_1 + \frac{\partial D_1}{\partial t} \\ J_2 &= j_2 + \frac{\partial D_2}{\partial t} \\ J_3 &= j_3 + \frac{\partial D_3}{\partial t} \end{aligned} \quad \rightarrow \quad \mathbf{J} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \tag{1.1}$$

$$\begin{aligned} \mu\alpha &= \frac{dH}{dy} - \frac{dG}{dz} \\ \mu\beta &= \frac{dF}{dz} - \frac{dH}{dx} \\ \mu\gamma &= \frac{dG}{dx} - \frac{dF}{dy} \end{aligned} \quad \rightarrow \quad \begin{aligned} \mu H_1 &= \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \\ \mu H_2 &= \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \\ \mu H_3 &= \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{aligned} \quad \rightarrow \quad \mu\mathbf{H} = \nabla \times \mathbf{A} \tag{1.2}$$

$$\begin{aligned} \frac{d\gamma}{dy} - \frac{d\beta}{dz} &= 4\pi p' \\ \frac{d\alpha}{dz} - \frac{d\gamma}{dx} &= 4\pi q' \\ \frac{d\beta}{dx} - \frac{d\alpha}{dy} &= 4\pi r' \end{aligned} \quad \rightarrow \quad \begin{aligned} \frac{\partial H_3}{\partial y} - \frac{\partial H_2}{\partial z} &= 4\pi J_1 \\ \frac{\partial H_1}{\partial z} - \frac{\partial H_3}{\partial x} &= 4\pi J_2 \\ \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} &= 4\pi J_3 \end{aligned} \quad \rightarrow \quad \nabla \times \mathbf{H} = \mathbf{J} \tag{1.3}$$

$$\begin{aligned} P &= \mu\left(\gamma\frac{dy}{dt} - \beta\frac{dz}{dt}\right) - \frac{dF}{dt} - \frac{d\Psi}{dx} \\ Q &= \mu\left(\alpha\frac{dz}{dt} - \gamma\frac{dx}{dt}\right) - \frac{dG}{dt} - \frac{d\Psi}{dy} \\ R &= \mu\left(\beta\frac{dx}{dt} - \alpha\frac{dy}{dt}\right) - \frac{dH}{dt} - \frac{d\Psi}{dz} \end{aligned} \quad \rightarrow \quad \begin{aligned} E_1 &= \mu\left(H_3 v_2 - H_2 v_3\right) - \frac{dA_1}{dt} - \frac{d\varphi}{dx} \\ E_2 &= \mu\left(H_1 v_3 - H_3 v_1\right) - \frac{dA_2}{dt} - \frac{d\varphi}{dy} \\ E_3 &= \mu\left(H_2 v_1 - H_1 v_2\right) - \frac{dA_3}{dt} - \frac{d\varphi}{dz} \end{aligned} \quad \rightarrow \quad \mathbf{E} = \mu\left(\mathbf{v} \times \mathbf{H}\right) - \frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi \tag{1.4}$$

$$\begin{aligned} P &= k\, f \\ Q &= k\, g \\ R &= k\, h \end{aligned} \quad \rightarrow \quad \begin{aligned} \varepsilon E_1 &= D_1 \\ \varepsilon E_2 &= D_2 \\ \varepsilon E_3 &= D_3 \end{aligned} \quad \rightarrow \quad \varepsilon\mathbf{E} = \mathbf{D} \tag{1.5}$$

$$\begin{aligned} P &= -\zeta p \\ Q &= -\zeta q \\ R &= -\zeta r \end{aligned} \quad \rightarrow \quad \begin{aligned} \sigma E_1 &= j_1 \\ \sigma E_2 &= j_2 \\ \sigma E_3 &= j_3 \end{aligned} \quad \rightarrow \quad \sigma\mathbf{E} = \mathbf{j} \tag{1.6}$$

$$e + \frac{d f}{dx} + \frac{d g}{dy} + \frac{d h}{dz} = 0 \quad \rightarrow \quad \rho + \frac{\partial D_1}{\partial x} + \frac{\partial D_2}{\partial y} + \frac{\partial D_3}{\partial z} = 0 \quad \rightarrow \quad \rho + \nabla \bullet \mathbf{D} = 0 \tag{1.7}$$

$$\frac{de}{dt} + \frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} = 0 \quad \rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial j_1}{\partial x} + \frac{\partial j_2}{\partial y} + \frac{\partial j_3}{\partial z} = 0 \quad \rightarrow \quad \frac{\partial \rho}{\partial t} + \nabla \bullet \mathbf{j} = 0 \tag{1.8}$$

These original equations do not strictly correspond with today's vector equations. The original equations, for example, contains the vector potential **A**, which usually is eliminated.

Three Maxwell equations can be identified quickly as OHM's law (1.6), the FARADAY-force (1.4) and the continuity equation (1.8) for a region with electric charges.

The Original Quaternion Form of Maxwell's Equations

In his Treatise[16] of 1873 MAXWELL has already modified his original equations of 1865. In addition Maxwell tried to introduce the quaternion notation by writing down his results also in a quaternion form. However, he has never really calculated with quaternions but only uses either the scalar or the vector part of a quaternion in his equations.

A general quaternion has a scalar (real) and a vector (imaginary) part. In the example below ‚a' is the scalar part and '*i*b + *j*c + *k*d' is the vector part.

$$\mathbb{Q} = a + ib + jc + kd$$

Here a, b, c and d are real numbers and i, j, k are the so-called HAMILTON'ian[7] unit vectors with the magnitude of √-1. They fulfill the equations

$$i^2 = j^2 = k^2 = ijk = -1$$

and

$$ij = k \quad jk = i \quad ki = j$$
$$ij = -ji \quad jk = -kj \quad ki = -ik$$

A nice presentation about the rotation capabilities of the HAMILTON'ian unit vectors in a three-dimensional ARGAND diagram was published by GOUGH[6].

Now MAXWELL has defined the field vectors (for example $\mathbf{B} = B_1 i + B_2 j + B_3 k$) as quaternions without scalar part and scalars as quaternions without vector part. In addition he defined a quaternion operator without scalar part

$$\nabla = \frac{d}{dx_1} i + \frac{d}{dx_2} j + \frac{d}{dx_3} k \quad ,$$

which he used in his equations. Maxwell separated a single quaternion with two prefixes into a scalar and vector part. He defined this prefixes as

$$S.\mathbb{Q} = S.(a + ib + jc + kd) = a$$

$$V.\mathbb{Q} = V.(a + ib + jc + kd) = ib + jc + kd$$

The original Maxwell quaternion equations are now for isotropic media (no changes except fonts, normal letter = scalar, capital letter = quaternion without scalar):

$$B = V.\nabla A \tag{1.9}$$

$$E = V.vB - \dot{A} - \nabla\Psi \tag{1.10}$$

$$F = V.vB + eE - m\nabla\Omega \tag{1.11}$$

$$B = H + 4\pi M \tag{1.12}$$

$$4\pi J_{tot} = V.\nabla H \tag{1.13}$$

$$J = CE \tag{1.14}$$

$$D = \frac{1}{4\pi} KE \tag{1.15}$$

$$J_{tot} = J + \dot{D} \tag{1.16}$$

$$B = \mu H \tag{1.17}$$

$$e = S.\nabla D \tag{1.18}$$

$$m = S.\nabla M \tag{1.19}$$

$$H = -\nabla\Omega \tag{1.20}$$

Beside the new notation, the magnetic potential field Ω and the magnetic mass m was mentioned here the first time. By calculating the gradient of this magnetic potential field it is possible to get the magnetic field (or in analogy the magnetostatic field. Maxwell has introduced this two new field variables into the force equation (1.11).

The reader may check that the equations above identical to the previous published equations (1.1) to (1.7), except the continuity equation (1.8) has this time be dropped. From the above notation it is clearly visible why the quaternion despite the deep engagement for example of Professor Peter Guthrie TAIT[19] did not succeed, then the new introduced vector notation of Oliver HEAVISIDE[11] and Josiah Willard GIBBS[23] was much easier to read and to use for most applications.

It is very interesting that Maxwell‘s first formulation of a magnetic charge density and the related discussion about the possible existence of magnetic monopoles became forgotten for more than half a century until Paul André Maurice DIRAC[4] again speculated about magnetic monopoles in 1931.

Today's Vector Notation of MAXWELL's Equations

The nowadays most often used notation can be easily derived from the original equations of 1865. By inserting (1.1) in (1.3) it follows the known equation

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j} \tag{1.21}$$

Equation (1.4) contains the FARADAY equation

$$\mathbf{E} = \mathbf{v} \times \mu \mathbf{H} \quad \xrightarrow{\mu = \text{konstant}} \quad \mathbf{E} = \mu \left(\mathbf{v} \times \mathbf{H} \right) \tag{1.22}$$

and the potential equation for the electric field

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \varphi \ . \tag{1.23}$$

Together with the potential equation for the magnetic field (1.2) follows with applying the rotation on both sides of (1.23)

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\left(\nabla \times \mathbf{A} \right) = \frac{\partial}{\partial t}\left(\mu \mathbf{H} \right) \quad \xrightarrow{\mu = \text{konstant}} \quad \nabla \times \mathbf{E} = \mu \frac{\partial \mathbf{H}}{\partial t} \tag{1.24}$$

From (1.2) follows further with the divergence:

$$\nabla \bullet \mu \mathbf{H} = 0 \quad \xrightarrow{\mu = \text{konstant}} \quad \mu \nabla \bullet \mathbf{H} = 0 \tag{1.25}$$

The six MAXWELL equations in today's notation are:

FARADAY's law
$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j} \tag{1.26}$$

AMPÈRE's law
$$-\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \tag{1.27}$$

COULOMB's law
$$\nabla \bullet \mathbf{D} = \rho \tag{1.28}$$

$$\nabla \bullet \mathbf{B} = 0 \tag{1.29}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \varepsilon_r \mathbf{E} = \varepsilon \mathbf{E} \tag{1.30}$$

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M} = \mu_0 \mu_r \mathbf{H} = \mu \mathbf{H} \tag{1.31}$$

with

E:	electrical field strength	[V / m]
H:	magnetic field strength	[A / m]
D:	electric displacement	[As / m^2]
B:	magnetic Induction	[Vs / m^2]
j:	electric current density	[A / m^2]
ε:	electric permeability	[As / Vm]
μ:	magnetic permeability	[Vs / Am]

Please note that the MAXWELL equation of today have became subset of the original equations which in turn have got an expansion with the introduction of the magnetic induction (1.31).

Today traditionally not included in MAXWELL's equations are FARADAY's law and sometime also OHM's law. The continuity equation (1.8) is rarely mentioned in literature. But this equation defines the conservation of charge:

$$\nabla\bullet\left(\nabla\times\mathbf{H}\right)=\frac{\partial}{\partial t}\left(\nabla\bullet\mathbf{D}\right)+\nabla\bullet\mathbf{j}=\frac{\partial\rho}{\partial t}+\nabla\bullet\mathbf{j}=0 \tag{1.32}$$

The electric and magnetic field strengths are interpreted as physically existent force fields, which are able to describe forces between electric and magnetic poles. Maxwell has – analogue to fluid mechanics – this force fields associated with two underlying potential fields, which are not shown anymore in the today's traditional vector notation. The force fields can be derived from the potential fields as:

$$-\mathbf{E}=\nabla\varphi+\frac{\partial\mathbf{A}}{\partial t} \tag{1.33}$$

$$\mathbf{B}=\nabla\times\mathbf{A} \tag{1.34}$$

with

φ:	electric potential field	[V]
A:	vector potential	[Vs / m]

For a very long time scientists are convinced that the potentials do not have any physical existence but merely are a mathematical construct. But an experiment suggested by Yakir AHARONOV and David BOHM[1] has shown, that this is not true. There arises the question about the causality of the fields. Many reasons point out that the potentials φ and **A** really are the cause of the force fields **E** and **H**.

Including the material equations (1.30) and (1.31), and with consideration of Ohm's law

$$\mathbf{j}=\sigma\mathbf{E} \tag{1.35}$$

with

σ:	specific electric conductivity	[1 / Ωm] = [A / Vm]

the Maxwell equations become for homogenous and isotropic conditions (ε = constant, μ = constant):

$$\nabla\times\mathbf{H}=\varepsilon\frac{\partial\mathbf{E}}{\partial t}+\sigma\mathbf{E} \tag{1.36}$$

$$-\nabla\times\mathbf{E}=\mu\frac{\partial\mathbf{H}}{\partial t} \tag{1.37}$$

$$\varepsilon\nabla\bullet\mathbf{E}=\rho \tag{1.38}$$

$$\mu\nabla\bullet\mathbf{H}=0 \tag{1.39}$$

Real Expansions of Maxwell‘s Equations

The HERTZ-Ansatz

Recently Thomas PHIPPS[20] has shown that Heinrich Rudolf HERTZ has suggested another possibility to adapt Maxwell‘s equations. During Hertz life this was hardly criticized and his proposal was vastly forgotten after his death. Usually the differentials are partial derivative and not total derivatives as shown in the comparison (1.1) to (1.8) between MAXWELL‘s original equations and the today‘s vector notation. Now in the equations (1.26) and (1.27) HERTZ has substituted the partial derivatives ∂ with the total derivatives d. With this Hertz tried to treat Maxwell equations invariant to the GALILEI-transformation:

$$\nabla \times \mathbf{H} = \frac{\mathrm{d}\mathbf{D}}{\mathrm{dt}} + \mathbf{j} \tag{1.40}$$

$$-\nabla \times \mathbf{E} = \frac{\mathrm{d}\mathbf{B}}{\mathrm{dt}} \tag{1.41}$$

what wit the entity $\frac{\mathrm{d}}{\mathrm{dt}} = \frac{\partial}{\partial \mathrm{t}} + \mathbf{v} \bullet \nabla$ becomes

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial \mathrm{t}} + \mathbf{v} \bullet \nabla \mathbf{D} + \mathbf{j} \tag{1.42}$$

$$-\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial \mathrm{t}} + \mathbf{v} \bullet \nabla \mathbf{B} \tag{1.43}$$

Now the question arises about the meaning of the newly introduced velocity **v**. HERTZ has interpreted this velocity as the (absolute) motion of aether elements. But if v is interpreted as relative velocity between charges, then Maxwell‘s equations are defined for the case v = 0, hat can be interpreted that the test charge does not move in the observer‘s reference frame. Therefore Thomas PHIPPS explains this velocity as the velocity of a test charge relative to an observer.

Consequently in equation (1.33) the partial derivatives have to be replaced wit the total derivatives, too.

$$-\mathbf{E} = \nabla \varphi + \frac{\mathrm{d}\mathbf{A}}{\mathrm{dt}} = \nabla \varphi + \frac{\partial \mathbf{A}}{\partial \mathrm{t}} + \mathbf{v} \bullet \nabla \mathbf{A} \tag{1.44}$$

The invariance of (1.40) and (1.41) against a GALILEI-transformation for the case that no current densities **j** and no charges are present can easily be seen. For **v** = 0 (a relative to the observer stationary charge) always MAXWELL‘s equations will be the result:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial \mathrm{t}} \tag{1.45}$$

$$-\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial \mathrm{t}} \tag{1.46}$$

For a GALILEI-transformation is **r‘** = **r** – **v**t and t‘ = t; thus for **v** > 0 is:

$$\frac{\partial}{\partial \mathrm{x}'} = \frac{\partial}{\partial \mathrm{x}} \quad \frac{\partial}{\partial \mathrm{y}'} = \frac{\partial}{\partial \mathrm{y}} \quad \frac{\partial}{\partial \mathrm{z}'} = \frac{\partial}{\partial \mathrm{z}} \quad \rightarrow \quad \nabla' = \nabla \qquad \text{and} \qquad \frac{\partial}{\partial \mathrm{t}'} = \frac{\partial}{\partial \mathrm{t}} + \mathbf{v} \bullet \nabla$$

from which for all **v** the equations

$$\nabla \times \mathbf{H} = \frac{d\mathbf{D}}{dt} \tag{1.47}$$

$$-\nabla \times \mathbf{E} = \frac{d\mathbf{B}}{dt}$$

are valid. If the observer moves together with a test charge, this reduces again to the equations (1.45) and (1.46). The first EINSTEIN postulate[5] sais, that in an uniform moving system all physical laws take its simplest form independent of the velocity. That is fulfilled in the example above. In each uniform moving reference frame the observer always measures for example the undamped wave equation.

The DIRAC-Ansatz

The non-symmetry in MAXWELL's equation system always has motivated to extend this set of equations. The most famous extension has originated form DIRAC [3], who suggested the following extension:

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{j}_e \tag{1.48}$$

$$-\nabla \times \mathbf{E} = \mu \frac{\partial \mathbf{H}}{\partial t} + \mathbf{j}_m \tag{1.49}$$

$$\varepsilon \nabla \bullet \mathbf{E} = \rho_e \tag{1.50}$$

$$\mu \nabla \bullet \mathbf{H} = \rho_m \tag{1.51}$$

Together with (1.51) this ansatz must lead to the postulation of magnetic monopoles, which until today never has been (absolutely certain) detected. As a consequence of this ansatz the force fields **E** and **B** are derived from potentials according to:

$$\mathbf{E} = \nabla \varphi - \frac{\partial \mathbf{A}}{\partial t} - \nabla \times \mathbf{C} \tag{1.52}$$

$$\mathbf{E} = \nabla \phi - \frac{\partial \mathbf{C}}{\partial t} - \nabla \times \mathbf{A} \tag{1.53}$$

where ϕ and **C** represent the complementary magnetic potentials. Therefore as another consequence there must exist two different kinds of photons, which interact in different ways with matter[14]. Also this has until today never been observed.

The HARMUTH-Ansatz

Henning HARMUTH[5] (and Konstantin MEYL[17])have gone a step further and suggested new equations, which differ to the DIRAC ansatz only in that point, that no source fields exists anymore. Harmuth has used this proposition to solve the problem of propagation of electromagnetic impulses in lossy media (impulses in media with low OHM dissipation) for the boundary conditions E = 0 and H = 0 for $t \leq 0$:

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E} \tag{1.54}$$

$$-\nabla \times \mathbf{E} = \mu \frac{\partial \mathbf{H}}{\partial t} + s\mathbf{H} \tag{1.55}$$

$$\varepsilon \nabla \bullet \mathbf{E} = 0 \tag{1.56}$$

$$\mu \nabla \bullet \mathbf{H} = 0 \tag{1.57}$$

with

s: specific magnetic conductivity [V / Am]

In his interpretation of this ansatz, MEYL has gone again a step further and declares the equations (1.54) to (1.57) to be valid in all cases. As a consequence there would exist no kind of monopoles, whether electric nor magnetic... The alleged electric monopoles (charges) are – according to Meyl – only secondary effects of electric or magnetic dipoles.

From (1.54) to (1.57) HARMUTH[5]-GL.21 has derived the electric field equation to

$$\Delta \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} - (\mu\sigma + \varepsilon s) \frac{\partial \mathbf{E}}{\partial t} - s\sigma \mathbf{E} = 0 \tag{1.58}$$

and has shown, that this equation can be solved for a certain set of boundary conditions. The same equation (1.58) is designated by Meyl as the fundamental field equation.

The MÚNERA-GUZMÁN-Ansatz

Héctor MÚNERA and Octavio GUZMÁN[19] have proposed the following equations ($\omega \equiv ct$):

$$\nabla \times \mathbf{N} = \frac{\partial \mathbf{P}}{\partial \omega} + \frac{4\pi}{c} \mathbf{J} \tag{1.59}$$

$$\nabla \times \mathbf{P} = -\frac{\partial \mathbf{N}}{\partial \omega} + \frac{4\pi}{c} \mathbf{J} \tag{1.60}$$

$$\nabla \bullet \mathbf{N} = 4\pi\rho \tag{1.61}$$

$$\nabla \bullet \mathbf{P} = -4\pi\rho \tag{1.62}$$

with

$$\mathbf{N} \equiv \mathbf{B} - \mathbf{E} \tag{1.63}$$

$$\mathbf{P} \equiv \mathbf{B} + \mathbf{E} \tag{1.64}$$

From this follows MAXWELL's equations (1.26)-(1.29) as shown below:

FARADAY's law (1.26): (1.60) – (1.59) (1.65)

AMPÈRE's law (1.27): (1.60) + (1.59) (1.66)

COULOMB's law (1.28): (1.62) – (1.61) (1.67)

(1.62) + (1.61) (1.68)

In this notation the current density J and the charge density ρ are not understood as electric only but merely as electromagnetic entities. With an analysis of MÚNERA and GUZMÁN it can be shown, that beneath the electric scalar field also a non-trivial magnetic scalar field should exist.

Imaginary Expansions of Maxwell's Equations

The Notation in Minkowski-Space

In electrodynamics the relativistic notation is fully established. Because of the second EINSTEIN'ian postulate[5] about the absolute constancy of the speed of light (therefore its independency of the speed of the light source or light detector) the four-dimensional notation has been developed. But the force field vectors **E** and **H** can not be used for four-vectors. But the potentials and the charge densities have been regarded as very optimal to formulate the electrodynamics in a compact form. If we first have an event vector

$$\mathbf{X} = \mathrm{ict} + \mathbf{x}$$

then it follows in MINKOWSKI-Space the invariance of the four-dimensional length ds

$$\mathrm{ds} = \sqrt{\mathrm{dx}_\mu \mathrm{dx}_\mu} = \sqrt{\mathrm{dx}_1\mathrm{dx}_1 + \mathrm{dx}_2\mathrm{dx}_2 + \mathrm{dx}_3\mathrm{dx}_3 - \mathrm{c}^2\mathrm{dt}^2}$$

and

$$\mathrm{d}\tau = \frac{\mathrm{i}}{\mathrm{c}}\sqrt{\mathrm{dx}_\mu \mathrm{dx}_\mu} = \sqrt{\mathrm{dt}^2 - \frac{1}{\mathrm{c}^2}\left(\mathrm{dx}_1\mathrm{dx}_1 + \mathrm{dx}_2\mathrm{dx}_2 + \mathrm{dx}_3\mathrm{dx}_3\right)}$$

From this follows the four-dimensional velocity vector to

$$\mathbf{U} = \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - \frac{\mathrm{u}^2}{\mathrm{c}^2}}}\left(\mathrm{ic} + \mathbf{u}\right)$$

which gives the four-dimensional current density

$$\mathbf{J} = \rho_0\mathbf{U} = \frac{\rho_0}{\sqrt{1 - \frac{\mathrm{u}^2}{\mathrm{c}^2}}}\left(\mathrm{ic} + \mathbf{u}\right)$$

With the four-dimensional gradient operator

$${}^4\nabla = \frac{\partial}{\mathrm{ic}\partial\mathrm{t}} + \frac{\partial}{\partial\mathrm{x}_1}\mathbf{i} + \frac{\partial}{\partial\mathrm{x}_2}\mathbf{j} + \frac{\partial}{\partial\mathrm{x}_3}\mathbf{k}$$

follows with

$${}^4\nabla \bullet \mathbf{J} = 0$$

the continuity equation (1.8). With the four-dimensional vector potential

$$\mathbf{A} = \mathrm{i}\varphi + \mathbf{A}$$

and with the D'ALEMBERT operator

$$\left({}^4\nabla\right)^2 = \Box^2 = \nabla^2 - \frac{1}{\mathrm{c}^2}\frac{\partial}{\partial\mathrm{t}^2} = \frac{\partial^2}{\partial\mathrm{x}_\nu\partial\mathrm{x}_\nu} - \frac{1}{\mathrm{c}^2}\frac{\partial}{\partial\mathrm{t}^2}$$

follows the relation

$$\Box^2\mathbf{A} = \frac{1}{\mathrm{c}}\mathbf{J}$$

But then the possibility for a compact and easy calculation within the MINKOWSKI-Space comes to an end. To include the electric and magnetic fields, the following definition

$$F_{\mu\nu} \equiv \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$$

is used to determine the electromagnetic field tensor:

$$\mathbf{F} = \begin{Bmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{Bmatrix}$$

With two equations with the components of the field tensor the four MAXWELL equations can be derived. With the first equation

$$\frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0$$

follows for an arbitrary combination of λ, μ, ν to 1, 2, 3 the MAXWELL equation (1.29)

$$\frac{\partial F_{12}}{\partial x_3} + \frac{\partial F_{23}}{\partial x_1} + \frac{\partial F_{31}}{\partial x_2} = 0 \quad \rightarrow \quad \frac{\partial B_3}{\partial x_3} + \frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} = \nabla \bullet \mathbf{B} = 0$$

and if one of the indices λ, μ, ν is equal 4 it follows the MAXWELL equation (1.27). With the second equation

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{1}{c}\mathbf{J}_\mu$$

follow the non-homogenous MAXWELL equations (1.26) and (1.28).

Simple Complex Notation

One possibility to enhance the symmetry of MAXWELL's equations offers the inclusion of imaginary numbers. INOMATA[13] uses the imaginary axis only for the „missing" terms in MAXWELL's equations. Thus they become:

$$\begin{aligned} &\mathbf{D} = \varepsilon\mathbf{E} \qquad \nabla \bullet \mathbf{D} = \rho \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j} \\ &\mathbf{B} = \mu\mathbf{H} \qquad \nabla \bullet \mathbf{B} = i\rho_m \qquad -\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} + i\,\mathbf{j}_m \end{aligned} \tag{1.69}$$

From this result an imaginary magnetic charge and an imaginary magnetic current density. In this notation the imaginary unit i is used for variables, which are not physically existent (i.e. are not measurable until now). Thus by using „i" in the equations above the missing variables are placed into an imaginary (non existent) space $i(x_1, x_2, x_3)$.

Eight-dimensional, Complex Notation

Elizabeth RAUSCHER[19] proposes a consequent expansion of the complex notation, so that for each field and for each charge density a real and an imaginary part is introduced.

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_{Re} + \mathrm{i}\,\mathbf{E}_{Im}\\ \mathbf{B} &= \mathbf{B}_{Re} + \mathrm{i}\,\mathbf{B}_{Im}\\ \mathbf{j} &= \mathbf{j}_{Re} + \mathbf{j}_{Im} = \mathbf{j}_e + \mathbf{j}_m\\ \rho &= \rho_{Re} + \rho_{Im} = \rho_e + \rho_m\end{aligned} \tag{1.70}$$

Then, when using a correct splitting of the terms, two complementary sets of MAXWELL equations can be formulated. The real equations are:

$$\begin{aligned}&\mathbf{D}_{Re} = \varepsilon\mathbf{E}_{Re} \qquad \nabla\bullet\mathbf{D}_{Re} = \rho_{Re} \qquad \nabla\times\mathbf{H}_{Re} = \frac{\partial\mathbf{D}_{Re}}{\partial \mathrm{t}} + \mathbf{j}_{Re}\\ &\mathbf{B}_{Re} = \mu\mathbf{H}_{Re} \qquad \nabla\bullet\mathbf{B}_{Re} = 0 \qquad -\nabla\times\mathbf{E}_{Re} = \frac{\partial\mathbf{B}_{Re}}{\partial \mathrm{t}}\end{aligned} \tag{1.71}$$

With an elimination of i on both sides we get for the imaginary parts:

$$\begin{aligned}&\mathbf{D}_{Im} = \varepsilon\mathbf{E}_{Im} \qquad \nabla\bullet\mathbf{D}_{Im} = 0 \qquad \nabla\times\mathbf{H}_{Im} = \frac{\partial\mathbf{D}_{Im}}{\partial \mathrm{t}}\\ &\mathbf{B}_{Im} = \mu\mathbf{H}_{Im} \qquad \nabla\bullet\mathbf{B}_{Im} = \rho_{Im} \qquad -\nabla\times\mathbf{E}_{Im} = \frac{\partial\mathbf{B}_{Im}}{\partial \mathrm{t}} + \mathbf{j}_{Im}\end{aligned} \tag{1.72}$$

As used by INOMATA also RAUSCHER uses the imaginary unit „i“ to sort the physical existent variables from the physical non existent ones.

The Imaginary Quaternion Notation

An other possibility is the mixture of quaternions and imaginary numbers, what has for example be done by HONIG[12]. With the vector potential and the current density

$$\begin{aligned}\mathbf{A}_4 &= \mathrm{i}\,\varphi + \mathrm{A_x}\,i + \mathrm{A_y}\,j + \mathrm{A_z}\,k\\ \mathbf{J}_4 &= \mathrm{i}\,\rho + \rho\mathrm{v_x}\,i + \rho\mathrm{v_y}\,j + \rho\mathrm{v_z}\,k\end{aligned} \tag{1.73}$$

follows with the operator

$$\square_{\mathrm{q}} = \mathrm{i}\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}}\,i + \frac{\partial}{\partial \mathrm{x}}\,j + \frac{\partial}{\partial \mathrm{x}}\,k \tag{1.74}$$

and with the LORENTZ condition $\square_{\mathrm{q}}\bullet\mathrm{A}_4 = 0$ the MAXWELL equations with

$$\square_{\mathrm{q}}^2\,\mathbf{A}_4 = \nabla\bullet\mathrm{i}\mathbf{E} + \nabla\bullet\mathbf{B} + \nabla\times\mathrm{i}\mathbf{E} + \nabla\times\mathbf{B} + \left(-\frac{\partial\mathbf{E}}{\partial \mathrm{t}} + \mathrm{i}\frac{\partial\mathbf{B}}{\partial \mathrm{t}}\right) = \mathrm{i}\rho + \mathbf{J}_3 = \mathbf{J}_4 \tag{1.75}$$

Actually this notation is very efficient. It is, for example, easily possible to formulate the LORENTZ force or equations of the quantum electrodynamics with this notation. But now the imaginary unit „i“ is not used to separate the observable variables from the non existent ones. Interestingly there does not exists one single real number at all. Each real number is associated either with the imaginary unit „i“ or with the HAMILTON units *i*, *j* and *k*. With some additional rules also this notation can be expanded to eight dimensions. This should be presented in another paper.

Closing Remarks

Different notations to the MAXWELL equations are presented. Depending on the application one or another notation can be very useful, but at the end the presented variety is not satisfactory.

Many discussions have been presented about the existence of magnetic monopoles. But either the electric field is only a subjective measuring caused by the relative motion between charges –as it is said by the Special Theory of Relativity – or the magnetic force field can be derived from a scalar potential field. In the first case magnetic monopoles can not exist, in the second case they can exist. Despite of extensive experiments no magnetic monopoles have been found until now. So we can conclude, that no magnetic potential fields must be postulated and that the non symmetry in MAXWELL's equations still are correct.

Proposals to enhance the symmetry with imaginary numbers are interesting but covers the danger, that with the simple mathematical tool „i" a symmetric formulation can be reached vastly, but that the physical models do become nebulous.

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